GROUP THEORY 2024 - 25, SOLUTION SHEET 8

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2.
(1) $1 \triangleleft \mathbb{Z}$ is a subnormal series with abelian quotients;
(2) $1 \triangleleft \mathbb{Z}/2\mathbb{Z} \triangleleft S_3$ is a subnormal series with abelian quotients;
(3) exercise 4 of last week;
(4) exercise 7 of last week;
(5) That's immediate by proposition 26.

Exercise 3. Observe that the derived group of a product of groups is the product of the derived groups. More generally if $H' \leq H$ and $K' \leq K$ are subgroups, then $[H' \times K', H \times K] = [H', H] \times [K', K]$. This can be proved easily by showing that the generators of each side are contained in the other. With the usual notations, we then get by induction that $(H \times G)^{\{i\}} = H^{\{i\}} \times G^{\{i\}}$ and thus this sequence becomes the trivial group at index $\max(n, m)$, where n and m are the indices when $H^{\{i\}}$ and $G^{\{i\}}$ become the trivial group, respectively.

Exercise 4.
(1) Note that conjugation by any element of the group G gives us an automorphism of G. Hence, if H is a characteristic subgroup of G, then $gHg^{-1} = H$, for all $g \in G$, so H is normal.
(2) Firstly, observe that if ϕ is an automorphism of G, then $\phi(Z(G)) \subset Z(G)$. Indeed, pick $a \in Z(G)$ and $g \in G$ arbitrary. We want to show that $\phi(a)g = g\phi(a)$ to conclude by arbitrary choice of a and g. As ϕ is an automorphism, it is in particular surjective and thus there exists $h \in G$ such that $\phi(h) = g$. We infer that

$$\phi(a)g = \phi(a)\phi(h) = \phi(ah) = \phi(ha) = \phi(h)\phi(a) = g\phi(a)$$

and we are done. For the reverse inclusion apply the same reasoning to the inverse automorphism ϕ^{-1} to get $Z(G) = \phi(Z(G))$, i.e. $Z(G)$ is a characteristic subgroup of G.
(3) Again, observe that if ϕ is an automorphism of G, we have $\phi([G, G]) \subset [G, G]$. Indeed, pick a commutator $[a, b] \in [G, G]$ and notice that $\phi([a, b]) = [\phi(a), \phi(b)] \in [G, G]$. Since the commutators generate $[G, G]$, we must have that $\phi([G, G]) \subseteq [G, G]$. Applying the same reasoning to the inverse automorphism ϕ^{-1} allows us to conclude.

Exercise 5. Since G is a finite p-group, we know by the lecture that $|G| = p^n$ for some $n \in \mathbb{N}$. We prove that G is solvable by induction on n. If $n = 1$ then $G \cong \mathbb{Z}/p\mathbb{Z}$ is solvable. If $n > 1$ we know that the center $Z(G)$ is non-trivial (exercise 2, sheet 4)), so that $G/Z(G)$ is a p-group of order p^k for some $k < n$. By induction this quotient is solvable. Since $Z(G)$ is abelian, it is solvable as well. It follows that G is solvable, being an extension of two solvable groups:

$$1 \to Z(G) \to G \to G/Z(G) \to 1.$$

Exercise 6. (1) As G is simple, any subnormal series of G must be given by $1 < G$ and because G is solvable, $G/1 \cong G$ must thus be abelian. As G is simple and abelian, G must be cyclic of prime order.

(2) By definition, each composition factor in a composition series of G is simple. Since each factor is a quotient of a subgroup of G, it must be solvable. Consequently, by the previous point, each composition factor is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. There is a finite number of composition factors, say $\mathbb{Z}/p_1\mathbb{Z}, \dots, \mathbb{Z}/p_n\mathbb{Z}$ for some n, and hence $|G| = \prod_{i=1}^{n}(p_i)$.

(3) " $\Longleftarrow$ " Trivial, because in this case the composition series precisely tells us that G is solvable.

" $\Longrightarrow$ " By the same argument used in (1) each of the composition factors must be solvable and simple, thus cyclic of prime order.

Exercise 7. To show that the group B of 2×2 invertible upper-triangular matrices over a field k is solvable, we need to construct a subnormal series for B where each quotient is abelian. We define a normal subgroup $U \triangleleft B$ by

$$U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in k \right\}.$$

This subgroup U is clearly isomorphic to the additive group $(k, +)$ which is abelian. Moreover U is normal in B by a straightforward computation.

We claim that the following forms a subnormal series for B with abelian quotients:

$$\{I\} \triangleleft U \triangleleft B.$$

Since U is abelian, we only need to observe that B/U is abelian. Define a map $B \to k^\times \times k^\times$ by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d)$. This is a surjective group homomorphism whose kernel is precisely U. By the first isomorphism theorem $B/U \cong k^\times \times k^\times$ which is abelian. This concludes the proof.

Exercise 8. (1) Let $\alpha \in S_n$ be a non trivial element and let $\alpha = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k$ be its decomposition into disjoint (non-identity) cycles. Write $\alpha_1 = (a_1, a_2, \dots, a_m)$ with $m \geq 2$. If $m \geq 3$, let $\beta = (a_1, a_2)$ and observe that

$$\alpha^{-1}\beta\alpha = (\alpha(a_1), \alpha(a_2)) = (\alpha_1(a_1), \alpha_1(a_2)) = (a_2, a_3) \neq \beta.$$

Hence $\alpha\beta \neq \beta\alpha$ and so α is not in the center of S_n. Now if $m = 2$ the cycle is $\alpha_1 = (a_1, a_2)$, and let $\beta = (a_1, a_2, a_3)$ for some $a_3 \neq a_1, a_2$. Then observe that

$$\alpha^{-1}\beta\alpha = (\alpha(a_1), \alpha(a_2), \alpha(a_3)) = (\alpha_1(a_1), \alpha_1(a_2), \alpha(a_3)) = (a_2, a_1, b)$$

for $b = \alpha(a_3)$ different from a_1 and a_2. Hence $\alpha^{-1}\beta\alpha \neq \beta$ which proves that α is not in the center of S_n.

(2) Let $1 \neq \sigma \in H$ be a non-trivial element. Since $H \cap A_n = 1$, we know that σ can be written as an odd product of transpositions and so $\sigma^2 \in A_n$. Since $\sigma^2 \in H$ as well, this proves that $\sigma^2 = 1$. Now if $\tau \in H$ is another non-trivial element, it can also be written as an odd product of transpositions and so $\sigma\tau \in A_n$. This implies that $\sigma\tau = 1$, so $\tau = \sigma^{-1} = \sigma$ and thus $H = \{1, \sigma\}$.

(3) Since the center of S_n is trivial, there must exists $\tau \in S_n$ such that $\tau^{-1}\sigma\tau \neq \sigma$. But since H is normal in S_n we have that $\tau^{-1}\sigma\tau \in H = \{1, \sigma\}$ and hence $\tau^{-1}\sigma\tau = 1$. This implies that $\sigma = 1$ and contradicts the fact that σ is non-trivial.

(4) Since $H \triangleleft S_n$, we have that $H \cap A_n \triangleleft A_n$. We have seen in class that A_n is simple and so either $H \cap A_n = 1$, either $H \cap A_n = A_n$. The first case is ruled out by the previous two points. In the second case, this implies that $A_n \subseteq H \subset S_n$. By the correspondence theorem, H corresponds to a subgroup of $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$. Since this group has no non-trivial subgroup, we obtain that either $H = A_n$ or $H = S_n$ as desired.